The Completeness Theorem of First-Order Logic

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1 First-Order Logic

1.1 Syntax

Definition 1.1.1 (First-order language)**.** A *first-order language* $\mathcal{L}$ is given by

- a set $\mathcal{F}_\mathcal{L}$ of *function symbols* with an arity function $\sharp\colon \mathcal{F}_\mathcal{L} \to \mathbb{N}$, and
- a set $\mathcal{R}_\mathcal{L}$ of *relation symbols* with an arity function $\sharp\colon \mathcal{R}_\mathcal{L} \to \mathbb{N}$,

where $\mathcal{F}_\mathcal{L} \cap \mathcal{R}_\mathcal{L} = \emptyset$.

We also always implicitly assume that $\mathcal{F}_\mathcal{L}$ and $\mathcal{R}_\mathcal{L}$ are disjoint form all other syntactical symbols we introduce. This can always be achieved by an appropriate disjoint-sum construction.

We use $\mathtt{f}$, $\mathtt{g}$, $\mathtt{h}$, … to denote function symbols and $\mathtt{R}$, $\mathtt{S}$, … to denote relation symbols.

Notation 1.1.2. For $\mathcal{L}$ a first-order language we use the notations $\mathcal{F}_\mathcal{L}^{(n)} = \{\mathtt{f} \in \mathcal{F}_\mathcal{L} \mid \sharp(\mathtt{f}) = n\}$ and $\mathcal{R}_\mathcal{L}^{(n)} = \{\mathtt{R} \in \mathcal{R}_\mathcal{L} \mid \sharp(\mathtt{R}) = n\}$.

We assume a countable infinite set Var of first-order variables. Elements of Var are denoted by $\mathtt{x}$, $\mathtt{y}$, $\mathtt{z}$, …

Definition 1.1.3 ($\mathcal{L}$-terms)**.** For a first-order language $\mathcal{L}$ we define the set of $\mathcal{L}$-terms inductively as follows.

- Every $\mathtt{x} \in \mathrm{Var}$ is an $\mathcal{L}$-term.
- If $\mathtt{f} \in \mathcal{F}_\mathcal{L}^{(n)}$ and $t_1, \dots t_n$ are $\mathcal{L}$-terms, then so is $\mathtt{f}t_1 \dots t_n$.

We use s, t, … to denote terms.

Lemma 1.1.4. *If $s_1, \dots, s_n, t_1, \dots t_n$ are $\mathcal{L}$-terms such that $s_1 \dots s_n = t_1 \dots t_n$, then $s_i = t_i$ for $1 \le i \le n$.*

Proof. Induction on the number of symbols of $s_1 \dots s_n$. The claim is trivial for $n = 0$, so let $n \geq 1$. If s_1 is a Variable, then t_1 has to be the same variable (as we assumed variables to be disjoint from the function symbols), and the claim follows by induction hypothesis. Otherwise, there are $\mathtt{f}$, $\mathtt{g}$, $s'_1, \dots, s'_k$, and $t'_1, \dots, t'_\ell$ such that $s_1 = \mathtt{f}s'_1 \dots s'_k$ and $t_1 = \mathtt{g}t'_1 \dots t'_\ell$. So $\mathtt{f}s'_1 \dots s'_k s_2 \dots s_n = \mathtt{g}t'_1 \dots t'_\ell t_2 \dots t_n$. Hence $\mathtt{f} = \mathtt{g}$, hence $k = \sharp(\mathtt{f}) = \sharp(\mathtt{g}) = \ell$ and the claim follow by induction hypothesis since $s'_1 \dots s'_k s_2 \dots s_n = t'_1 \dots t'_\ell t_2 \dots t_n$. ∎

Lemma 1.1.5. *Every $\mathcal{L}$-term t is either a variable or there exist unique $\mathtt{f}$, $t_1, \dots, t_n$ such that $t = \mathtt{f}t_1 \dots t_n$.*

Proof. Existence follows from the definition of terms. As for uniqueness, assume $\mathtt{f}t_1 \dots t_n = \mathtt{g}s_1 \dots s_m$. Then $\mathtt{f} = \mathtt{g}$, hence $n = \sharp(\mathtt{f}) = \sharp(\mathtt{g}) = m$ and the claim follows from Lemma 1.1.4. ∎

In particular, even though we think of terms as sequences of symbols, we can do induction on the build-up in their inductive definition.

Definition 1.1.6 ($\mathcal{L}$-formulae)**.** The set of $\mathcal{L}$-formulae is inductively defined as follows.

- If $\mathtt{R} \in \mathcal{R}_{\mathcal{L}}^{(n)}$ and $t_1, \dots t_n$ are $\mathcal{L}$-terms, then $\mathtt{R}t_1 \dots t_n$ and $\underline{\neg}\mathtt{R}t_1 \dots t_n$ are $\mathcal{L}$-formulae.
- If A and B are $\mathcal{L}$-formualae, then so are $\underline{\wedge}AB$ and $\underline{\vee}AB$.
- If A is an $\mathcal{L}$-formula and $\mathtt{x}$ is a variable, the $\underline{\forall}\mathrm{x}A$ and $\underline{\exists}\mathrm{x}A$ are $\mathcal{L}$-formulae.

We denote formulae by A, B, C, … As in Lemma 1.1.5 we can show unique readability of formulae.

Definition 1.1.7 ($\neg A$)**.** By induction of the definition of A we define a formula $\neg A$ as follows.

- $\neg(\mathtt{R}t_1 \dots t_n) = \underline{\neg}\mathtt{R}t_1 \dots t_n$ and $\neg(\underline{\neg}\mathtt{R}t_1 \dots t_n) = \mathtt{R}t_1 \dots t_n$
- $\neg(\underline{\wedge}AB) = \underline{\vee}(\neg A)(\neg B)$ and $\neg(\underline{\vee}AB) = \underline{\wedge}(\neg A)(\neg B)$.
- $\neg(\underline{\forall}\mathrm{x}A) = \underline{\exists}\mathrm{x}(\neg A)$ and $\neg(\underline{\exists}\mathrm{x}A) = \underline{\forall}\mathrm{x}(\neg A)$.

Remark 1.1.8. It holds that $\neg(\neg A) = A$.

Proof. Induction on A. ∎

We also use the abbreviations $A \wedge B$, $A \vee B$, $\exists \mathrm{x}A$, and $\forall \mathrm{x}A$ for $\underline{\wedge}AB$, $\underline{\vee}AB$, $\underline{\exists}\mathrm{x}A$, and $\underline{\forall}\mathrm{x}A$, respectively. Moreover, we use the abbreviation $A \to B$ for $(\neg A) \vee B$.

1.2 Semantics

Definition 1.2.1 ($\mathcal{L}$-structure)**.** Let $\mathcal{L}$ be a first-order language. An $\mathcal{L}$-structure $\mathfrak{M}$ is given by

- a non-empty set $|\mathfrak{M}|$, called the *universe of* $\mathfrak{M}$,
- a function $\mathtt{f}^{\mathfrak{M}} \colon |\mathfrak{M}|^{\sharp(\mathtt{f})} \to |\mathfrak{M}|$ for every function symbol $\mathtt{f}$ of $\mathcal{L}$, and
- a set $\mathtt{R}^{\mathfrak{M}} \subseteq |\mathfrak{M}|^{\sharp(\mathtt{R})}$ for every relation symbol $\mathtt{R}$ of $\mathcal{L}$.

Definition 1.2.2 ($\mathfrak{M}$-valuation)**.** If $\mathfrak{M}$ is structure, an $\mathfrak{M}$-valuation is a function $\xi \colon \mathrm{Var} \to |\mathfrak{M}|$.

Definition 1.2.3 ($t^{\mathfrak{M},\xi}$)**.** For t an $\mathcal{L}$-term, $\mathfrak{M}$ an $\mathcal{L}$-structure, and ξ an $\mathfrak{M}$-valuation we define $t^{\mathfrak{M},\xi} \in |\mathfrak{M}|$ by induction on t as follows.

$$\begin{array}{lcl} \mathtt{x}^{\mathfrak{M},\xi} & = & \xi(\mathtt{x}) \\ (\mathtt{f}t_1 \dots t_n)^{\mathfrak{M},\xi} & = & \mathtt{f}^{\mathfrak{M}}({t_1}^{\mathfrak{M},\xi}, \dots, {t_n}^{\mathfrak{M},\xi}) \end{array}$$

Definition 1.2.4 ($\xi^a_{\mathtt{x}}$)**.** For ξ an $\mathfrak{M}$-valuation, $\mathtt{x} \in \mathrm{Var}$, and $a \in |\mathfrak{M}|$ we define an $\mathfrak{M}$-valuation $\xi^a_{\mathtt{x}}$ as follows.

$$\xi^a_{\mathtt{x}}(\mathtt{y}) = \begin{cases} a & \mathtt{x} = \mathtt{y} \\ \xi(\mathtt{y}) & \text{otherwise} \end{cases}$$

Definition 1.2.5 ($A^{\mathfrak{M},\xi}$)**.** For A an $\mathcal{L}$-formula, $\mathfrak{M}$ an $\mathcal{L}$-structure and ξ an $\mathfrak{M}$-valuation we define $A^{\mathfrak{M},\xi} \in \{0,1\}$ by induction on A as follows.

$$\begin{array}{lcl} (\mathtt{R}t_1 \dots t_n)^{\mathfrak{M},\xi} & = & \begin{cases} 1 & ({t_1}^{\mathfrak{M},\xi}, \dots, {t_n}^{\mathfrak{M},\xi}) \in \mathtt{R}^{\mathfrak{M}} \\ 0 & \text{otherwise} \end{cases} \\ (\underline{\neg}\mathtt{R}t_1 \dots t_n)^{\mathfrak{M},\xi} & = & \begin{cases} 1 & ({t_1}^{\mathfrak{M},\xi}, \dots, {t_n}^{\mathfrak{M},\xi}) \notin \mathtt{R}^{\mathfrak{M}} \\ 0 & \text{otherwise} \end{cases} \end{array}$$

$$\begin{array}{lcl} (\underline{\wedge}AB)^{\mathfrak{M},\xi} & = & \min\{A^{\mathfrak{M},\xi}, B^{\mathfrak{M},\xi}\} \\ (\underline{\vee}AB)^{\mathfrak{M},\xi} & = & \max\{A^{\mathfrak{M},\xi}, B^{\mathfrak{M},\xi}\} \\ (\underline{\forall}\mathrm{x}A)^{\mathfrak{M},\xi} & = & \min\{A^{\mathfrak{M},\xi^a_{\mathtt{x}}} \mid a \in |\mathfrak{M}|\} \\ (\underline{\exists}\mathrm{x}A)^{\mathfrak{M},\xi} & = & \max\{A^{\mathfrak{M},\xi^a_{\mathtt{x}}} \mid a \in |\mathfrak{M}|\} \end{array}$$

Definition 1.2.6 ($\mathfrak{M}, \xi \models A$)**.** We write $\mathfrak{M}, \xi \models A$ to denote $A^{\mathfrak{M},\xi} = 1$.

Remark 1.2.7. Immediately from the definition we note the following.

- $\mathfrak{M}, \xi \models A \wedge B$ if and only if $\mathfrak{M}, \xi \models A$ and $\mathfrak{M}, \xi \models B$.
- $\mathfrak{M}, \xi \models A \vee B$ if and only if $\mathfrak{M}, \xi \models A$ or $\mathfrak{M}, \xi \models B$.
- $\mathfrak{M}, \xi \models \forall \mathrm{x} A$ if and only if for all $a \in |\mathfrak{M}|$ it holds that $\mathfrak{M}, \xi^a_{\mathtt{x}} \models A$.

- $\mathfrak{M}, \xi \models \exists \mathtt{x} A$ if and only if for some $a \in |\mathfrak{M}|$ it holds that $\mathfrak{M}, \xi^a_{\mathtt{x}} \models A$.

Lemma 1.2.8. *$\mathfrak{M}, \xi \models \neg A$ holds if and only if $\mathfrak{M}, \xi \models A$ does not hold.*

Proof. Induction on A. ∎

Definition 1.2.9 ($\mathfrak{M}, \xi \models \Theta$)**.** If Θ is a set of $\mathcal{L}$-formulae, $\mathfrak{M}$ an $\mathcal{L}$-structure and ξ an $\mathfrak{M}$-valuation, we write $\mathfrak{M}, \xi \models \Theta$ to denote that for all $A \in \Theta$ it holds that $\mathfrak{M}, \xi \models A$.

Definition 1.2.10. A set Θ of $\mathcal{L}$-formulae is said to *h*ave a model, if there is an $\mathcal{L}$-structure $\mathfrak{M}$ and an $\mathfrak{M}$-valuation ξ such that $\mathfrak{M}, \xi \models \Theta$.

Definition 1.2.11 ($\Theta \models A$)**.** If Θ is a set of $\mathcal{L}$-formulae, and A an $\mathcal{L}$-formula, we write $\Theta \models A$ to denote that for all $\mathcal{L}$-structures $\mathfrak{M}$ and $\mathfrak{M}$-valuations ξ for which $\mathfrak{M}, \xi \models \Theta$ holds, it also holds that $\mathfrak{M}, \xi \models A$.

2 The Tait Calculus

2.1 Derivations

Definition 2.1.1. For an $\mathcal{L}$-term t we define a finite set $\mathrm{FV}(t) \subseteq \mathrm{Var}$, the *set of free variables of* t, inductively as follows.

- $\mathrm{FV}(\mathtt{x}) = \{\mathtt{x}\}$.
- $\mathrm{FV}(\mathtt{f} t_1 \ldots t_n) = \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$.

Definition 2.1.2. For an $\mathcal{L}$-formula A we define a finite set $\mathrm{FV}(A) \subseteq \mathrm{Var}$, the *set of free variables of* A, inductively as follows.

- $\mathrm{FV}(\mathtt{R} t_1 \ldots t_n) = \mathrm{FV}(\underline{\neg}\mathtt{R} t_1 \ldots t_n) = \mathrm{FV}(t_1) \cup \ldots \cup \mathrm{FV}(t_n)$
- $\mathrm{FV}(\underline{\wedge} AB) = \mathrm{FV}(\underline{\vee} AB) = \mathrm{FV}(A) \cup \mathrm{FV}(B)$
- $\mathrm{FV}(\underline{\forall}\mathtt{x} A) = \mathrm{FV}(\underline{\exists}\mathtt{x} A) = \mathrm{FV}(A) \setminus \{\mathtt{x}\}$

Definition 2.1.3. For a finite set Δ of $\mathcal{L}$-formulae, we define $\mathrm{FV}(\Delta) \subseteq \mathrm{Var}$, the *set of free variables of* Δ, by $\mathrm{FV}(\{A_1, \ldots, A_n\}) = \mathrm{FV}(A_1) \cup \ldots \cup \mathrm{FV}(A_n)$

Definition 2.1.4. We call a term, a formula, or a finite set of formulae *closed* if the set of free variables is empty.

Definition 2.1.5 ($s[t/\mathtt{x}]$)**.** For s and t terms, and $\mathtt{x}$ a variable we define a term $s[t/\mathtt{x}]$ by induction on s as follows.

- $\mathtt{y}[t/\mathtt{x}] = \begin{cases} t & \mathtt{x} = \mathtt{y} \\ \mathtt{y} & \mathtt{x} \neq \mathtt{y} \end{cases}$
- $(\mathtt{f} s_1 \ldots s_n)[t/\mathtt{x}] = \mathtt{f}(s_1[t/\mathtt{x}]) \ldots (s_n[t/\mathtt{x}])$

Definition 2.1.6 ($A[t/\mathtt{x}]$)**.** For A a formula, t a term, and $\mathtt{x}$ a variable we define a term $s[t/\mathtt{x}]$ by induction on A as follows.

- $(\mathtt{R}s_1 \dots s_n)[t/\mathtt{x}] = \mathtt{R}(s_1[t/\mathtt{x}]) \dots (s_n[t/\mathtt{x}])$
- $(\underline{\neg}\mathtt{R}s_1 \dots s_n)[t/\mathtt{x}] = \underline{\neg}\mathtt{R}(s_1[t/\mathtt{x}]) \dots (s_n[t/\mathtt{x}])$
- $(\underline{\wedge}AB)[t/\mathtt{x}] = \underline{\wedge}(A[t/\mathtt{x}])(B[t/\mathtt{x}])$
- $(\underline{\vee}AB)[t/\mathtt{x}] = \underline{\vee}(A[t/\mathtt{x}])(B[t/\mathtt{x}])$
- $(\underline{\forall}\mathtt{y}A)[t/\mathtt{x}] = \underline{\forall}\mathtt{z}(A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}])$, where $\mathtt{z}$ is the smallest variable such that $\mathtt{z} \notin \mathrm{FV}(\underline{\forall}\mathtt{y}A) \cup \mathrm{FV}(t) \cup \{\mathtt{x}\}$.
- $(\underline{\exists}\mathtt{y}A)[t/\mathtt{x}] = \underline{\exists}\mathtt{z}(A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}])$, where $\mathtt{z}$ is the smallest variable such that $\mathtt{z} \notin \mathrm{FV}(\underline{\exists}\mathtt{y}A) \cup \mathrm{FV}(t) \cup \{\mathtt{x}\}$.

Remark 2.1.7. Immediately from the definition of $A[t/\mathtt{x}]$, we note that the number of logical symbols is not changed by substitution.

Definition 2.1.8 (Proof symbols)**.** If $\mathcal{L}$ is a first order language, the language $\mathcal{T}_\mathcal{L}$ of proofs is given by the function symbols $\mathtt{I}_{\mathtt{R}t_1 \dots t_{\sharp(\mathtt{R})}}$, $\underline{\wedge}_{A,B}$, $\underline{\vee}_{A,B}$, $\underline{\forall}_{\mathtt{x},\mathtt{y},A}$, $\underline{\exists}_{\mathtt{x},t,A}$, and $\mathtt{C}_A$ where $\mathtt{R}$ is an $\mathcal{L}$-relation symbol, the t_i are $\mathcal{L}$-terms, and $\mathtt{x}$ is a variable, and A and B are $\mathcal{L}$-formulae.

Here $\sharp(\mathtt{I}_{\mathtt{R}t_1 \dots t_{\sharp(\mathtt{R})}}) = 0$, $\sharp(\underline{\wedge}_{A,B}) = \sharp(\mathtt{C}_A) = 2$, and $\sharp(\underline{\vee}_{A,B}) = \sharp(\underline{\forall}_{\mathtt{x},\mathtt{y},A}) = \sharp(\underline{\exists}_{\mathtt{x},t,A}) = 1$.

For Δ and Γ finite set of $\mathcal{L}$-formulae, and A an $\mathcal{L}$-formula, we write Δ, Γ as an abbreviation for $\Delta \cup \Gamma$ and Δ, A as an abbreviation for $\Delta \cup \{A\}$.

Definition 2.1.9 (Tait Calculus for first-order logic)**.** Given a first-order language $\mathcal{L}$, we inductively define the relation $d \vdash \Delta$ for d a closed $\mathcal{T}_\mathcal{L}$-term and Δ a finite set of $\mathcal{L}$-formulae as follows.

- $\mathtt{I}_{\mathtt{R}t_1 \dots t_{\sharp(\mathtt{R})}} \vdash \Delta, \mathrm{R}t_1 \dots t_{\sharp(\mathtt{R})}, \neg \mathrm{R}t_1 \dots t_{\sharp(\mathtt{R})}$
- If $d \vdash \Delta, A$ and $d' \vdash \Gamma, B$, then $\underline{\wedge}_{A,B} dd' \vdash \Delta, \Gamma, A \wedge B$.
- If $d \vdash \Delta, A, B$ then $\underline{\vee}_{A,B} d \vdash \Delta, A \vee B$.
- If $d \vdash \Delta, A[\mathtt{y}/\mathtt{x}]$ and $\mathtt{y} \notin \mathrm{FV}(\Delta, \underline{\forall}\mathrm{x}A)$, then $\underline{\forall}_{\mathtt{x},\mathtt{y},A} d \vdash \Delta, \forall \mathrm{x}A$.
- If $d \vdash \Delta, A[t/\mathtt{x}]$, then $\underline{\exists}_{\mathtt{x},t,A} d \vdash \Delta, \exists \mathrm{x}A$.
- If $d \vdash \Delta, A$ and $d' \vdash \Gamma, \neg A$, then $\mathtt{C}_A dd' \vdash \Delta, \Gamma$.

Proposition 2.1.10. *If $d \vdash \Delta$ and Δ' is obtained form Δ by consistent renaming of variables, then $d' \vdash \Delta'$ where d and d' only differ in naming of variables.*

Proof. Induction on the inductive Definition 2.1.9. In the case $\underline{\forall}_{\mathtt{x},\mathtt{y},A} d$ we pick a new $\mathtt{y}$ not occurring in the renamed context and obtain a renaming of the premise. In the case of $\underline{\exists}_{\mathtt{x},t,A} d$ we rename the witness t according to the renaming and use that substitution and renaming fit together. ∎

Proposition 2.1.11 (Weakening)**.** *If $d \vdash \Delta$, then $d' \vdash \Delta, \Gamma$ where d' is obtained from d by renaming of variables.*

Proof. Induction on the inductive Definition 2.1.9. In the case $\underline{\forall}_{\mathtt{x},\mathtt{y},A} d$ we might have to pick a new $\mathtt{y}$ if $\mathtt{y} \in \mathrm{FV}(\Gamma)$; this is possible as $\mathrm{FV}(\Gamma, \Delta)$ is finite. So we need a renaming of the proof obtained by induction hypothesis, which is provided by Proposition 2.1.10. ∎

Remark 2.1.12. By Proposition 2.1.11, we may, without loss of generality, assume that in Definition 2.1.9 the side formulas Δ and Γ are always the full set of formulas concluded (in particular, only new formulae are added when stepping from a proof to its subproofs). With this assumption, we can reconstruct the derived set of formulae for all subproofs in the inductive Definition 2.1.9 of $d \vdash \Delta$ from d and Δ.

Definition 2.1.13 ($\neg\Delta$)**.** If $\Delta = \{A_1, \ldots, A_n\}$ is a finite set of formulae, we define $\neg\Delta$ for $\{\neg A_1, \ldots, \neg A_n\}$.

Definition 2.1.14 ($\Theta \vdash \Delta$)**.** For Θ a set of formulae and Δ a finite set of formulae, we write $\Theta \vdash \Delta$, if for some finite $\Gamma \subseteq \Theta$ and some proof term d have $d \vdash \neg\Gamma, \Delta$.

Remark 2.1.15. Immediately from the definition we note that, if $\Theta \subseteq \Theta'$ and $\Theta \vdash \Delta$ then $\Theta' \vdash \Delta$.

Definition 2.1.16 ($\Theta \vdash A$)**.** For Θ a set of formulae and A a formula, we write $\Theta \vdash A$ for $\Theta \vdash \{A\}$.

Proposition 2.1.17 (Identity)**.** *There is a family I_A of proof terms such that $I_A \vdash A, \neg A$.*

Proof. Induction on A. We choose $I_{\mathtt{R}t_1 \ldots t_{\sharp(\mathtt{R})}} = \mathtt{I}_{\mathtt{R}t_1 \ldots t_{\sharp(\mathtt{R})}}$, $I_{A \wedge B} = \underline{\vee}_{\neg A, \neg B} \underline{\wedge}_{A,B} I_A I_B$, and $I_{\forall \mathtt{x} A} = \underline{\forall}_{\mathtt{x},\mathtt{x},A} \underline{\exists}_{\mathtt{x},\mathtt{x},\neg A} I_A$. ∎

Corollary 2.1.18. $A \vdash A$

Proposition 2.1.19. *If $\Theta \vdash A \wedge B$ then $\Theta \vdash A$ and $\Theta \vdash B$.*

Proof. Assume $d \vdash \neg\Gamma, A \wedge B$. Then $\mathtt{C}_{A \wedge B} d \underline{\vee}_{A,B} I_A \vdash \neg\Gamma, A$ and $\mathtt{C}_{A \wedge B} d \underline{\vee}_{A,B} I_B \vdash \neg\Gamma, B$. ∎

Proposition 2.1.20. *If $\Theta \vdash \forall \mathtt{x} A$ then $\Theta \vdash A[t/\mathtt{x}]$.*

Proof. Assume $d \vdash \neg\Gamma, \forall \mathtt{x} A$. Then $\mathtt{C}_{\forall \mathtt{x} A} d \underline{\exists}_{\mathtt{x},t,\neg A} I_{A[t/\mathtt{x}]} \vdash \neg\Gamma, A[t/\mathtt{x}]$. ∎

Proposition 2.1.21 (Modus ponens)**.** *If $\Theta \vdash A \to B$ and $\Theta \vdash A$ then $\Theta \vdash B$.*

Proof. Assume $d \vdash \neg\Gamma, A \to B$ and $d' \vdash \neg\Delta, A$. Then $\mathtt{C}_A d' \mathtt{C}_{A \to B} d \underline{\wedge}_{A,\neg B} I_A I_B \vdash \neg\Gamma, \neg\Delta, B$. ∎

2.2 Soundness

Lemma 2.2.1. *For all variables* $\mathtt{x}$*, all* $\mathcal{L}$*-terms* s *and* t*, all* $\mathcal{L}$*-structures* $\mathfrak{M}$ *and all* $\mathfrak{M}$*-valuations* ξ *the following holds.*

$$(s[t/\mathtt{x}])^{\mathfrak{M},\xi} = s^{\mathfrak{M},\xi_{\mathtt{x}}^{t^{\mathfrak{M},\xi}}}$$

Proof. Induction on s. ∎

Lemma 2.2.2 (Coincidence)**.** *Let* ξ *and* ξ' *be* $\mathfrak{M}$*-valuations such that for all* $\mathtt{x} \in \mathrm{FV}(t)$ *it holds that* $\xi(\mathtt{x}) = \xi'(\mathtt{x})$*. Then* $t^{\mathfrak{M},\xi} = t^{\mathfrak{M},\xi'}$

Proof. Induction on t. ∎

Lemma 2.2.3 (Coincidence)**.** *Let* ξ *and* ξ' *be* $\mathfrak{M}$*-valuations such that for all* $\mathtt{x} \in \mathrm{FV}(A)$ *it holds that* $\xi(\mathtt{x}) = \xi'(\mathtt{x})$*. Then* $A^{\mathfrak{M},\xi} = A^{\mathfrak{M},\xi'}$

Proof. Induction on A. The only non-trivial case is $\underline{\forall}\mathtt{x}A$, the case $\underline{\exists}\mathtt{x}A$ is similar.

Indeed, $(\underline{\forall}\mathtt{x}A)^{\mathfrak{M},\xi} = \min\{A^{\mathfrak{M},\xi^a_{\mathtt{x}}} \mid a \in |\mathfrak{M}|\}$, so it is enough to show that $\xi^a_{\mathtt{x}}$ and $\xi'^a_{\mathtt{x}}$ coincide on $\mathrm{FV}(A) \subseteq \mathrm{FV}(\underline{\forall}\mathtt{x}A) \cup \{\mathtt{x}\}$ for all a. But since $\xi^a_{\mathtt{x}}(\mathtt{x}) = a = \xi'^a_{\mathtt{x}}(\mathtt{x})$ this follows from the premise. ∎

Lemma 2.2.4. *For all variables* $\mathtt{x}$*, all* $\mathcal{L}$*-terms* t*, all* $\mathcal{L}$*-formulae* A*, all* $\mathcal{L}$*-structures* $\mathfrak{M}$ *and all* $\mathfrak{M}$*-valuations* ξ *the following holds.*

$$(A[t/\mathtt{x}])^{\mathfrak{M},\xi} = A^{\mathfrak{M},\xi_{\mathtt{x}}^{t^{\mathfrak{M},\xi}}}$$

Proof. Induction on the number of logical symbols in A, keeping all other parameters universally quantified. The only non-trivial case is $\underline{\forall}\mathtt{x}A$, the case $\underline{\exists}\mathtt{x}A$ is similar.

We have $(\underline{\forall}\mathtt{y}A)[t/\mathtt{x}]^{\mathfrak{M},\xi} = \underline{\forall}\mathtt{z}(A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}])^{\mathfrak{M},\xi} = \min\{A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}]^{\mathfrak{M},\xi^a_{\mathtt{z}}} \mid a \in |\mathfrak{M}|\}$. By Induction hypothesis (compare Remark 2.1.7), we obtain $A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}]^{\mathfrak{M},\xi^a_{\mathtt{z}}} = A[\mathtt{z}/\mathtt{y}]^{\mathfrak{M},\xi^a_{\mathtt{z}}{}^{t^{\mathfrak{M},\xi^a_{\mathtt{z}}}}_{\mathtt{x}}} = A^{\mathfrak{M},\xi^a_{\mathtt{z}}{}^{t^{\mathfrak{M},\xi^a_{\mathtt{z}}}}_{\mathtt{x}}{}^{a}_{\mathtt{y}}}$ where we used $\xi^a_{\mathtt{z}}{}^{t^{\mathfrak{M},\xi^a_{\mathtt{z}}}}_{\mathtt{x}}(\mathtt{z}) = a$ since $\mathtt{z} \neq \mathtt{x}$. By coincidence for terms, we have $t^{\mathfrak{M},\xi^a_{\mathtt{z}}} = t^{\mathfrak{M},\xi}$, as $\mathtt{z} \notin \mathrm{FV}(t)$. Hence $A[\mathtt{z}/\mathtt{y}][t/\mathtt{x}]^{\mathfrak{M},\xi^a_{\mathtt{z}}} = \cdots = A^{\mathfrak{M},\xi^a_{\mathtt{z}}{}^{t^{\mathfrak{M},\xi}}_{\mathtt{x}}{}^{a}_{\mathtt{y}}} = A^{\mathfrak{M},\xi^{t^{\mathfrak{M},\xi}}_{\mathtt{x}}{}^{a}_{\mathtt{y}}}$ where for the last equation we used that, if $\mathtt{z} = \mathtt{y}$ the valuations are equal and if $\mathtt{z} \neq \mathtt{y}$, then $\mathtt{z} \notin \mathrm{FV}(A) \subseteq \mathrm{FV}(\underline{\forall}\mathtt{y}A) \cup \{\mathtt{y}\}$ and the equality follows from coincidence for formulae. The claim follows. ∎

Theorem 2.2.5 (Soundness)**.** *Let* $\mathcal{L}$ *be a first-order language. If* $d \vdash A_1, \ldots, A_n$ *for* $\mathcal{L}$*-formulae* $A_1, \ldots, A_n$*, then, for all* $\mathcal{L}$*-structures* $\mathfrak{M}$ *and all* $\mathfrak{M}$*-valuations* ξ *there is an* $1 \leq i \leq n$ *such that* $\mathfrak{M}, \xi \models A_i$*.*

Proof. By induction on d.

Case $\underline{\exists}_{\mathtt{x},t,A}d$. Let $\mathfrak{M}$ and ξ be given. If $\mathfrak{M}, \xi \models A[t/\mathtt{x}]$, then, by Lemma 2.2.4, $\mathfrak{M}, \xi^{t^{\mathfrak{M},\xi}}_{\mathtt{x}} \models A$, hence $\mathfrak{M}, \xi \models \underline{\exists}\mathtt{x}A$.

Case $\underline{\forall}_{\mathtt{x},\mathtt{y},A}d$. Let $\mathfrak{M}$ and ξ be given. If for all $a \in |\mathfrak{M}|$ we have $\mathfrak{M}, \xi^a_{\mathtt{y}} \models A[\mathtt{y}/\mathtt{x}]$, then we note $A[\mathtt{y}/\mathtt{x}]^{\mathfrak{M},\xi^a_{\mathtt{y}}} = A^{\mathfrak{M},\xi^a_{\mathtt{y}}{}^{a}_{\mathtt{x}}} = A^{\mathfrak{M},\xi^a_{\mathtt{x}}}$ since $\mathtt{y} \notin \mathrm{FV}(\underline{\forall}\mathtt{x}A)$. Hence

$\mathfrak{M}, \xi \models \forall \mathrm{x} A$. If, on the other hand, for some $a \in |\mathfrak{M}|$ is not the case that $\mathfrak{M}, \xi^a_{\mathtt{y}} \models A[\mathtt{y}/\mathrm{x}]$, then $\mathfrak{M}, \xi^a_{\mathtt{y}} \models A_i$ for some of side formula A_i. But then also $\mathfrak{M}, \xi \models A_i$ since $\mathtt{y} \notin A_i$ and the claim follows.

The remaining cases are simple. ∎

Corollary 2.2.6. *If $\Theta \vdash A$ then $\Theta \models A$.*

Proof. Assume $d \vdash \neg B_1, \ldots, \neg B_n, A$ for some $\{B_1, \ldots, B_n\} \subseteq \Theta$. Let $\mathfrak{M}$ and ξ with $\mathfrak{M}, \xi \models \Theta$ be given. We have to show $\mathfrak{M}, \xi \models A$. By Theorem 2.2.5 we have $\mathfrak{M}, \xi \models A$ or $\mathfrak{M}, \xi \models \neg B_i$ for some i. The latter, however, is impossible (Lemma 1.2.8) since we have $\mathfrak{M}, \xi \models \Theta$, hence $\mathfrak{M}, \xi \models B_i$. ∎

3 The Completeness Theorem

3.1 Maximally consistent sets

Definition 3.1.1. A set Θ of formulae is called *inconsistent*, if $\Theta \vdash \emptyset$. A set Θ is called *consistent*, if it is not inconsistent.

Remark 3.1.2. Immediately from the definition we note that a set is inconsistent if and only if some finite subset is.

Definition 3.1.3. A set Θ of $\mathcal{L}$-formulae is called a *maximally consistent set of $\mathcal{L}$-formulae*, if Θ is consistent, and for every set Θ' of $\mathcal{L}$-formulae with $\Theta' \supseteq \Theta$ that is also consistent, we have $\Theta' = \Theta$.

Definition 3.1.4. A set Θ of formulae *shows witnesses*, if for every formula $\exists \mathrm{x} A \in \Theta$ there is a term t such that $A[t/\mathrm{x}] \in \Theta$.

Lemma 3.1.5. *If Θ is maximally consistent and $\Theta \vdash A$, then $A \in \Theta$.*

Proof. Let $d \vdash \neg\Delta, A$ for some finite $\Delta \subseteq \Theta$. It is enough to show that Θ, A is consistent. If not, say $d' \vdash \neg\Delta', \neg A$ for some finite $\Delta' \subseteq \Theta$, then $\mathtt{C}_A d d' \vdash \neg\Delta, \neg\Delta'$ contradicting the consistency of Θ. ∎

Corollary 3.1.6. *Let Θ be maximally consistent.*

- *If $A[t/\mathrm{x}] \in \Theta$, then $\exists \mathrm{x} A \in \Theta$.*
- *If $A, B \in \Theta$, then $A \wedge B \in \Theta$.*
- *If $A \in \Theta$, then $A \vee B, B \vee A \in \Theta$.*

Lemma 3.1.7. *If Θ is maximally consistent and A a formula, then $A \in \Theta$ or $\neg A \in \Theta$.*

Proof. We have to show that at least one of Θ, A and $\Theta, \neg A$ is consistent. If not, we have proofs $d' \vdash \neg\Delta', \neg A$ and $d \vdash \neg\Delta, A$. But then $\mathtt{C}_A d d' \vdash \neg\Delta, \neg\Delta'$. ∎

Lemma 3.1.8. *Let Θ be a maximally consistent set of $\mathcal{L}$-formulae that shows witnesses, and let A be an $\mathcal{L}$-formula. Assume moreover, that for every $\mathcal{L}$-term t, we have $A[t/\mathtt{x}] \in \Theta$. Then $\forall \mathtt{x} A \in \Theta$.*

Proof. By Lemma 3.1.7 it is enough to exclude the case that $\exists \mathtt{x} \neg A \in \Theta$. Indeed, if $\exists \mathtt{x} \neg A \in \Theta$, then, as Θ shows witnesses, we have $\neg A[t/\mathtt{x}] \in \Theta$ for some term t. But since $A[t/\mathtt{x}] \in \Theta$ by assumption, Θ is inconsistent (Proposition 2.1.17). □

Lemma 3.1.9. *Let Θ be a maximally consistent set of $\mathcal{L}$-formulae that shows witnesses. Then there is an $\mathcal{L}$-structure $\mathfrak{M}$ and an $\mathfrak{M}$-valuation ξ such that for every $\mathcal{L}$-formula A we have $\mathfrak{M}, \xi \models A$ if and only if $A \in \Theta$. Moreover, $|\mathfrak{M}|$ is the set of $\mathcal{L}$-terms.*

Proof. We define $\mathfrak{M}$ as follows. Let $|\mathfrak{M}|$ be the set of all $\mathcal{L}$-terms and interpret the function symbols canonically, i.e., we set $\mathtt{f}^{\mathfrak{M}}(t_1, \ldots, t_{\sharp(\mathtt{f})}) = \mathtt{f} t_1 \ldots t_{\sharp(\mathtt{f})}$. We interpret relation symbols as given by Θ, i.e., we set $\mathtt{R}^{\mathfrak{M}} = \{(t_1, \ldots, t_{\sharp(\mathtt{R})}) \mid \mathtt{R} t_1 \ldots t_{\sharp(\mathtt{R})} \in \Theta\}$.

We define ξ as $\xi(\mathtt{x}) = \mathtt{x}$, i.e., ξ maps every variable to the term given by that variable. By induction on t we show that $t^{\mathfrak{M},\xi} = t$.

By induction on the number of logical symbols of A we show that if $\mathfrak{M}, \xi \models A$ then $A \in \Theta$. Indeed, the case of atomic formulae is given by the definition of the $\mathtt{R}^{\mathfrak{M}}$ and Lemma 3.1.7; the case of conjuction, disjunction, and existential quantification is given by Corollary 3.1.6; and the case of universal quantification is given by Lemma 3.1.8.

If, on the other hand, it is not the case that $\mathfrak{M}, \xi \models A$, then $\mathfrak{M}, \xi \models \neg A$ by Lemma 1.2.8. So, as just shown, $\neg A \in \Theta$. Therefore, $A \notin \Theta$ by Proposition 2.1.17 as Θ is consistent. □

Lemma 3.1.10. *Let Θ be a consistent set of $\mathcal{L}$-formulae. Then there is a set $\Theta' \supseteq \Theta$ of $\mathcal{L}$-formulae that is maximally consistent.*

Proof. Consider the collection of consistent set $\Theta' \supseteq \Theta$ of $\mathcal{L}$-formulae, ordered by inclusion. Obviously, this collection contains Θ and hence is not empty.

Consider a totally ordered subcollection; we claim that its union is consistent (and hence an upper bound in the collection). Indeed, if the union were inconsistent, then there would be finitely many formulae witnessing that inconsistency (Remark 3.1.2). But since the subcollection is totally ordered, there is a single element containing all those formulae, hence that set would be inconsistent, which it is not (as we only considered consistent sets).

By Zorn's Lemma, the collection has a maximal element. That is a maximally consistent set above Θ. □

Remark 3.1.11. The use of choice in Lemma 3.1.10 can be avoided if a well-ordering on $\mathcal{L}$ is given (e.g., if $\mathcal{L}$ is finite). That well-ordering extends to a well ordering on all $\mathcal{L}$-formulae. So, by transfinite recursion, we can consider each formula and add it, if the resulting set stays consistent (taking the union on limit stages).

Lemma 3.1.12. *Let c is an $\mathcal{L}$-constant, i.e., $c \in \mathcal{F}^{(0)}_{\mathcal{L}}$, that does not occur in any element of Θ, Δ. Moreover, assume $\Theta, A[c/\mathtt{x}] \vdash \Delta$. Then $\Theta, \exists \mathtt{x} A \vdash \Delta$.*

Proof. Assume $d \vdash \neg\Gamma, \neg A[c/\mathtt{x}], \Delta$ for some finite $\Gamma \subseteq \Theta$. First, we show by induction on derivations that replacing a constant by a new (for the whole derivation) variable yields again a valid derivation. Applying this general observation to d, we obtain $d' \vdash \neg\Gamma, \neg A[\mathtt{y}/\mathtt{x}], \Delta$ where $\mathtt{y}$ is a fresh variable. Hence $\underline{\forall}_{\mathtt{x},\mathtt{y},A} d' \vdash \neg\Gamma, \forall \mathtt{x} \neg A, \Delta$. ∎

Lemma 3.1.13. *If Θ is consistent, then so is $\Theta \cup \{A[c_{\mathtt{x},A}/\mathtt{x}] \mid \exists \mathtt{x} A \in \Theta\}$ where the $c_{\mathtt{x},A}$ are pairwise distinct constants not occurring in any element of Θ.*

Proof. Assume $d \vdash \neg\Gamma, \neg A_1[c_{\mathtt{x}_1,A_1}/\mathtt{x}_1], \dots, \neg A_k[c_{\mathtt{x}_k,A_k}/\mathtt{x}_k]$ for some finite $\Gamma \subseteq \Theta$ and $\exists \mathtt{x}_1 A_1, \dots, \exists \mathtt{x}_k A_k \in \Theta$. By induction on k, using Lemma 3.1.12, we obtain $d' \vdash \neg\Gamma, \neg(\exists \mathtt{x}_1 A_1), \dots, \neg(\exists \mathtt{x}_k A_k)$ for some d'. ∎

Remark 3.1.14. The set $\Theta \cup \{A[c_{\mathtt{x},A}/\mathtt{x}] \mid \exists \mathtt{x} A \in \Theta\}$ constructed in Lemma 3.1.13 does not necessarily show witnesses, as one of the newly added $A[c_{\mathtt{x},A}/\mathtt{x}]$ might be an existential formula itself.

Lemma 3.1.15. *Let Θ be a consistent set of $\mathcal{L}$ formulae. Then there exists a first-order language $\mathcal{L}'$ extending $\mathcal{L}$ and a maximally consistent set $\Theta' \supseteq \Theta$ of $\mathcal{L}'$-formulae that shows witnesses. Moreover, the cardinality of $\mathcal{L}'$ does not exceed the maximum of that of $\mathcal{L}$ and that of the natural numbers.*

Proof. We set $\Theta_0 = \Theta$ and $\mathcal{L}_0 = \mathcal{L}$. Then, by induction on n, take $\Theta'_{n+1} \supseteq \Theta_n$ a maximally consistent set of $\mathcal{L}_n$ formulae, let $\mathcal{L}_{n+1}$ be $\mathcal{L}_n$ extended by new constants $c^{n+1}_{\mathtt{x},A}$ for $\exists \mathtt{x} A \in \Theta'_{n+1}$, and set $\Theta_{n+1} = \Theta'_{n+1} \cup \{A[c^{n+1}_{\mathtt{x},A}/\mathtt{x}] \mid \exists \mathtt{x} A \in \Theta'_{n+1}\}$. Finally let $\mathcal{L}'$ be the union of all the $\mathcal{L}_n$ and set $\Theta' = \bigcup_n \Theta_n$.

Θ' shows witnesses. Indeed, if $\exists \mathtt{x} A \in \Theta'$, then $\exists \mathtt{x} A \in \Theta_n$ for some n. But then $A[c^{n+1}_{\mathtt{x},A}/\mathtt{x}] \in \Theta_{n+1} \subseteq \Theta'$.

Θ' is consistent by the same argument as in the proof of Lemma 3.1.10. We show that Θ' is a maximally consistent set of $\mathcal{L}'$-formulae. Let $\hat{\Theta} \supseteq \Theta'$ be a consistent set of $\mathcal{L}'$-formulae. We have to show $\hat{\Theta} = \Theta'$, i.e., $\hat{\Theta} \subseteq \Theta'$ (as the other inclusion holds by assumption). So let $A \in \hat{\Theta}$. We have to show $A \in \Theta'$. Since A is an $\mathcal{L}'$-formula, it is an $\mathcal{L}_n$-formula for some n. But then $A, \Theta'_{n+1} \supseteq \Theta'_{n+1}$ is a set of $\mathcal{L}_n$-formulae. Moreover, since $A, \Theta'_{n+1} \subseteq A, \Theta' \subseteq \hat{\Theta}$, the set A, Θ'_{n+1} is also consistent. Hence $A \in \Theta'_{n+1} \subseteq \Theta'$. ∎

Theorem 3.1.16. *If $\Theta \models A$, then $\Theta \vdash A$.*

Proof. Assume $\Theta \models A$, but not $\Theta \vdash A$. Then $\Theta, \neg A$ is a consistent set of $\mathcal{L}$-formulae. By Lemma 3.1.15 there is language $\mathcal{L}'$ extending $\mathcal{L}$ and a maximally consistent set $\Theta' \supseteq \Theta, \neg A$ that shows witnesses. Let $\mathfrak{M}$ and ξ be the $\mathcal{L}'$-structure and valuation given by Lemma 3.1.9. Then $\mathfrak{M}, \xi \models \Theta$. Moreover, $\mathfrak{M}$ can also be seen as an $\mathcal{L}$-structure by only interpreting symbols of $\mathcal{L}$; for $\mathcal{L}$-formulae, the notion of holding in the structure coincides (induction on the formula). Hence, since $\Theta \models A$, we have $\mathfrak{M}, \xi \models A$. But that implies $A \in \Theta'$, which is a contradiction since $\neg A \in \Theta'$ and Θ' is consistent (Proposition 2.1.17). ∎

3.2 Further Conclusions

Lemma 3.2.1. *A set Θ of $\mathcal{L}$-formulae has a model if and only if it is consistent.*

Proof. Assume Θ has a model, say $\mathfrak{M}, \xi \models \Theta$. We have to show that Θ is not inconsistent. So assume it were, say $d \vdash \neg A_1, \ldots, \neg A_n$ for some $A_1, \ldots, A_n \in \Theta$. Then, by soundness (Theoem 2.2.5) there is an $1 \leq i \leq n$ such that $\mathfrak{M}, \xi \models \neg A_i$, contradicting $\mathfrak{M}, \xi \models \Theta$.

Assume, on the other hand, that Θ is consistent. We have to show that Θ has a model. By Lemma 3.1.15 there is a first-order language $\mathcal{L}'$ extending $\mathcal{L}$ and a maximally consistent set $\Theta' \supseteq \Theta$ of $\mathcal{L}'$-formulae that shows witnesses. By Lemma 3.1.9 we obtain an $\mathcal{L}'$-structure $\mathfrak{M}$ and an $\mathfrak{M}$-valuation ξ such that $\mathfrak{M}, \xi \models \Theta$. Restricting the model to $\mathcal{L}$, we obtain an $\mathcal{L}$-model of Θ. ∎

Theorem 3.2.2 (Compactness)**.** *A set of $\mathcal{L}$-formulae has model, if and only if every finite subset has a model.*

Proof. Obviously, if the whole set has a model, then so has every subset. So let's assume that every finite subset has a model. We have to show that the whole set has a model. By Lemma 3.2.1 it is enough to show that the set is consistent. By Remark 3.1.2 it is enough to show that every finite subset is consistent. But this is the case, as it has a model. ∎

Theorem 3.2.3 (Löwenheim-Skolem)**.** *Let $\mathcal{L}$ be a first-order language. If a set of $\mathcal{L}$-formulae has a model, it also has one of size at most the maximum of that of $\mathcal{L}$ and that of the natural numbers.*

Proof. If a set Θ has a model, it is consistent. By Lemma 3.1.15 there is a first-order language $\mathcal{L}'$ extending $\mathcal{L}$ of the correct size and a maximally consistent set $\Theta' \supseteq \Theta$ of $\mathcal{L}'$-formulae showing witnesses. By Lemma 3.1.9 there is an $\mathcal{L}'$-model of the correct size. By restricting the signature, it can be seen as an $\mathcal{L}$-model. ∎

Example 3.2.4. The language of set theory is finite (one relation symbol $\in$, no function symbols). So, if there is a model of set theory, there is also a countable one.

Remark 3.2.5. Theorems 3.2.2 and 3.2.3 are purely model-theoretic statements, i.e., the statement of the theorem does not at all refer to a notion of proof.